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# An extension of the Avron–Herbst transformation to the quantized Schrödinger field for an interacting many-electron system

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**Abstract.** The effect of a uniform electric field on the time evolution of a single-electron wavefunction can be rigorously expressed by making use of the Avron–Herbst transformation. We extend the transformation to the quantized Schrödinger field for a many-electron system with an electron–electron interaction. The transformation for the time-ordered product of the Schrödinger field, which is the operator part of the Green function, is also derived.

## 1. Introduction

In condensed matter theory the electrical conductivity of electrons is one of the most important subjects and many theoretical attempts have been made [1, 2]. One of the widely used practical formulations is the linear-response theory, which gives physically acceptable results when it is used together with the finite temperature generalized Ward–Takahashi relations [3]. However, there are still unsolved fundamental difficulties in the theory of electrical conductivity. The main difficulties of the theory may be classified into two categories. One is the statistical physics aspect concerning the non-equilibrium and irreversible nature of the problem [4]. Another is the microscopic aspect, that is, the many-body wavefunction of electrons under the influence of the external electric field. Intensive research has been made by a number of authors on the former aspect of the problem. However, the latter does not seem to have attracted much attention despite its obvious significance in the theory of the electrical conductivity of electrons. The aim of this paper is to provide a new theoretical tool that seems to be useful for the investigation of the latter aspect.

When an external electric field is applied to an electron gas, its wavefunction cannot remain as a simple plane wave. From a classical mechanical point of view, the electrons will be accelerated and gain additional momentum and kinetic energy. The accelerated electrons may be scattered by impurities or lattice ions and again accelerated by the electric field. The number of repetitions of this process will be averaged and observed as a macroscopic steady state of electrical current [5]. In order to apply the quantum mechanical formulation to the problem, we need to know the behaviour of the electron wavefunction under the influence of an external electric field.

The effects of an external electric field on Bloch electrons have been studied by several authors [6–12]. Among them, Avron and Herbst [12] gave an important transformation

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for an electron wavefunction. For any wavefunction satisfying the free Schrödinger equation, the transformation yields the time evolution with an external uniform electric field. Their result seems to be of paramount significance in the investigation of the electron wavefunction in an electric field, because it provides a useful tool to construct a physically meaningful wavepacket that propagates under the influence of the electric field, without any approximation.

In metals or semiconductors, electrons cannot be described by a simple assembly of individual single-electron wavefunctions. We have to take into account the statistics of the particles. For electrons, the many-electron wavefunction must be antisymmetric under the permutation of electrons. Such antisymmetric many-electron wavefunctions can be adequately described in terms of the second quantized Schrödinger field. In this paper we shall present an extension of the Avron–Herbst transformation to the quantized Schrödinger field that describes an interacting many-electron system. We shall derive an extended Avron–Herbst transformation for the electron Schrödinger field operator and also the time-ordered product of the field operators. The transformation for the time-ordered product can be used to obtain a corresponding transformation for the Green function if it is used with a model ground state of the electron system.

In section 2 we consider a single-electron wavefunction and derive the Avron–Herbst transformation. In section 3 we consider the second quantized electron Schrödinger field and show that the Avron–Herbst transformation can be extended to the electron field with the electron–electron interaction. In section 4 the Avron–Herbst transformation is further extended to the time-ordered product of the electron Schrödinger field. Brief concluding remarks shall be given in section 5.

## 2. Avron–Herbst transformation

In this section we shall derive the Avron–Herbst transformation from a physical point of view, leaving the rigorous mathematical discussion to their original paper [12]. The Schrödinger equation for a particle in a uniform external field such as electric field or gravitational field has been discussed in the literature [13]. One of the possible solutions is the well known Airy function, whose asymptotic behaviour is fully understood [14]. Here we consider an electron in a uniform electric field. For simplicity we neglect the spin variables of the electron. In the following discussion the spin is not essential. The Schrödinger equation for the electron is

$$i\hbar \frac{\partial}{\partial t} \psi(\boldsymbol{x}, t) = \left( -\frac{\hbar^2}{2m} \nabla^2 - Fx \right) \psi(\boldsymbol{x}, t) \tag{2.1}$$

where $m$ is the electron mass, $\boldsymbol{x} \equiv (x, y, z)$ and $F = -eE$ with the electron charge $-e$. We assume that the electric field is along the $x$-axis, i.e. $\boldsymbol{E} = (E, 0, 0)$. Then the wavefunction can be factorized as

$$\psi(\boldsymbol{x}, t) = \varphi(x, t)\chi(y, z). \tag{2.2}$$

The equation for $\chi(y, z)$ is trivial. It simply gives a plane-wave solution. Our main concern is in the equation for $\varphi(x, t)$:

$$i\hbar \frac{\partial}{\partial t} \varphi(x, t) = \hat{h}_F(x, \partial_x)\varphi(x, t) \tag{2.3}$$

where the differential operator

$$\hat{h}_F(x, \partial_x) \equiv -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - Fx \tag{2.4}$$

has been used. A solution of equation (2.3) can be written as

$$\varphi(x, t) = \varphi_\varepsilon(x) \exp\left(\frac{i}{\hbar}\varepsilon t\right) \tag{2.5}$$

where $\varepsilon$ is an energy eigenvalue and $\varphi_\varepsilon(x)$ is the Airy function [13]

$$\varphi_\varepsilon(x) = \frac{1}{2\pi\sqrt{F}} \int_\Gamma dk \, \exp i\left\{\left(x + \frac{\varepsilon}{F}\right)k - \frac{\hbar^2}{6mF}k^3\right\}. \tag{2.6}$$

The integral contour $\Gamma$ is given in [13]. Eigenfunctions (2.5) form an orthonormal complete set:

$$\int_{-\infty}^{\infty} d\varepsilon \, \varphi_\varepsilon^*(x)\varphi_\varepsilon(x') = \delta(x - x') \tag{2.7a}$$

$$\int_{-\infty}^{\infty} dx \, \varphi_\varepsilon^*(x)\varphi_{\varepsilon'}(x) = \delta(\varepsilon - \varepsilon'). \tag{2.7b}$$

Having this set of eigenfunctions it is tempting to construct a Fock space on this basis and proceed to the many-electron quantum field theory. In order to have a physically meaningful Fock space, the basis states must have a clear particle nature, i.e. they must represent physically acceptable 'wavepacket' states. However, the physical meaning of the eigenfunction given by (2.5) is not simple. It expresses a standing wave. That is, the wavefunction contains travelling waves of both directions, yet it is not a bound state. Such eigenfunctions do not fit in with the realistic picture of the electrons in metals or semiconductors.

A possible picture of the electrons under the influence of the electric field can be given as follows. As an initial wavefunction of the electron we may assume a plane wave with a definite momentum, which shows that the electron is travelling in a certain direction. Then a spatially uniform electric field is adiabatically added. Consequently, the initial plane-wave state will be modified because of the electrical force acting on the electron. Hence, a physically relevant question is how such an acceleration process changes the wavefunction; namely, the time evolution of the initial plane wave. To obtain the time evolution of the wavefunction under the influence of the electric field we can use the spectral representation of the time-evolution operator in term of the Airy functions. This can be carried out by applying the time-evolution unitary operator to the plane wave,

$$\psi_p(x, t) = \exp\left[\frac{-i}{\hbar}t\hat{h}_F(x, \partial_x)\right]\exp(ipx) \tag{2.8}$$

and expanding the plane wave in terms of the Airy eigenfunctions:

$$\exp(ipx) = \frac{1}{2\pi\sqrt{F}} \int_{-\infty}^{\infty} d\varepsilon \, \exp i\left\{\frac{-\varepsilon}{F}p + \frac{\hbar^2}{6mF}p^3\right\}\varphi_\varepsilon(x). \tag{2.9}$$

Then the result of the time-evolution operator can be easily obtained. It yields

$$\psi_p(x, t) = \exp i\left\{\frac{\hbar^2}{6mF}\left[p^3 - \left(p + \frac{F}{\hbar}t\right)^3\right]\right\}\exp i\left\{\left(p + \frac{F}{\hbar}t\right)x\right\}. \tag{2.10}$$

To see the physical meaning of this result it is more convenient to define

$$u(x, p) \equiv \exp\left(\frac{-i\hbar^2}{6mF}p^3\right)\exp(ipx). \tag{2.11}$$

If the phase factor of the initial plane-wave state is chosen in this way, then the time evolution under the influence of the electric field is given as

$$\exp\left(\frac{-\mathrm{i}t}{\hbar}\hat{h}_F(x, \partial_x)\right) u(x, p) = u(x, p(t)) \tag{2.12}$$

where the time-dependent momentum variable is

$$p(t) = p + \frac{F}{\hbar}t. \tag{2.13}$$

That is, the momentum is increased by an amount of $Ft$ and the phase factor is also changed. It should be noted that the functional form of $u(x, p(t))$ defined by (2.11) does not change during the time evolution. Therefore, the wavefunction $u(x, p(t))$ seems to be suited for studying the effects of the applied electric field on the electrons.

The result (2.10) can be rewritten as

$$\psi_p(x, t) = \exp\left(\mathrm{i}\frac{F}{\hbar}tx - \mathrm{i}\frac{F^2}{6m\hbar}t^3\right) \exp\left(\frac{-\mathrm{i}}{\hbar}\frac{(\hbar p)^2}{2m}t\right) \exp\left[\mathrm{i}p\left(x - \frac{F}{2m}t^2\right)\right]$$

$$= \exp\{-\mathrm{i}\Lambda(x, t)\}\left[\exp\left(\frac{-\mathrm{i}}{\hbar}t\hat{h}_0(\nabla')\right)\exp(\mathrm{i}px')\right]_{x'=x-\frac{F}{2m}t^2} \tag{2.14}$$

where

$$\Lambda(x, t) = -\frac{F}{\hbar}tx + \frac{F^2}{6m\hbar}t^3 \tag{2.15}$$

and

$$\hat{h}_0(\partial_x) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{2.16}$$

have been defined. Equation (2.14) suggests that a solution of the free Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\varphi(x, t) = \hat{h}_0(\partial_x)\varphi(x, t) \tag{2.17}$$

can be transformed to

$$\psi(x, t) = \exp[-\mathrm{i}\Lambda(x, t)]\varphi\left(x - \frac{F}{2m}t^2, t\right) \tag{2.18}$$

which satisfies

$$\left[\mathrm{i}\hbar\frac{\partial}{\partial t} - \hat{h}_F(x, \partial_x)\right]\psi(x, t) = \left[\mathrm{e}^{-\mathrm{i}\Lambda(x,t)}\left\{\mathrm{i}\hbar\frac{\partial}{\partial t} - \hat{h}_0(\partial_x')\right\}\varphi(x', t)\right]_{x'=x-\frac{F}{2m}t^2}. \tag{2.19}$$

This result can be proved by a direct calculation. By virtue of (2.17), (2.19) yields

$$\left[\mathrm{i}\hbar\frac{\partial}{\partial t} - \hat{h}_F(x, \partial_x)\right]\psi(x, t) = 0. \tag{2.20}$$

Formula (2.18) transforms a solution of the free Schrödinger equation (2.17) to the case that has an external electric field, (2.20). Equation (2.18) is nothing more than the Avron–Herbst transformation [12].

To conclude this section let us remark on another formulation of the time evolution of the electron wavefunction in an external electric field. Roy and Mahapatra [11] gave an evolution operator for a crystal electron under the combined effect of an external electric field and a time-dependent electromagnetic field. Although the relation between their approach and the Avron–Herbst transformation is not clear at present, a first-order approximation of their evolution operator seems to confirm the validity of (2.13).

## 3. Electron Schrödinger field

Obviously, for studying a many-electron problem it is convenient to use the second quantized electron field [15–17]. In the nonrelativistic case, let us assume the Schrödinger field for the electrons. First we consider the electrons without an external electric field. These basic assumptions are that the electron Schrödinger field $\phi_\alpha(\boldsymbol{x})$ and $\Phi_\beta^\dagger(\boldsymbol{x}')$ satisfy the equal-time canonical anticommutation relation [15]

$$[\Phi_\alpha(\boldsymbol{x}), \Phi_\beta^\dagger(\boldsymbol{x})]_+ \equiv \Phi_\alpha(\boldsymbol{x})\Phi_\beta^\dagger(\boldsymbol{x}') + \Phi_\beta^\dagger(\boldsymbol{x}')\Phi_\alpha(\boldsymbol{x}) = \delta_{\alpha\beta}\delta(\boldsymbol{x} - \boldsymbol{x}') \qquad (3.1)$$

where $\alpha$ and $\beta$ denote the spin variables. Then the Hamiltonian of the electrons is assumed as

$$H = H_0 + H_{\text{int}} \qquad (3.2)$$

with

$$H_0 = \sum_\alpha \int \mathrm{d}^3 x \, \Phi_\alpha^\dagger(\boldsymbol{x}, t)\tilde{h}_0(\nabla)\Phi_\alpha(\boldsymbol{x}, t) \qquad (3.3)$$

and

$$H_{\text{int}} = \tfrac{1}{2}\sum_\alpha \sum_\beta \int \mathrm{d}^3 x \int \mathrm{d}^3 x' \, \Phi_\alpha^\dagger(\boldsymbol{x}) \, \Phi_\beta^\dagger(\boldsymbol{x}')V(|\boldsymbol{x} - \boldsymbol{x}'|)\Phi_\beta(\boldsymbol{x}')\Phi_\alpha(\boldsymbol{x}) \qquad (3.4)$$

where $V(|\boldsymbol{x} - \boldsymbol{x}'|)$ is an electron–electron interaction potential. In (3.3) we have used the kinetic-energy differential operator

$$\tilde{h}_0(\nabla) \equiv -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right). \qquad (3.5)$$

The Heisenberg picture for an arbitrary field operator $\Omega = \Omega(0)$ can be expressed in terms of the Hamiltonian as

$$\Omega(t) \equiv \exp\left(\frac{\mathrm{i}}{\hbar}Ht\right)\Omega(0)\exp\left(\frac{-\mathrm{i}}{\hbar}Ht\right). \qquad (3.6)$$

Then the equation of motion for the electron Schrödinger field is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Phi_\alpha(\boldsymbol{x}, t) = \tilde{h}_0(\nabla)\Phi_\alpha(\boldsymbol{x}, t) + \sum_\beta \int \mathrm{d}^3 x'' \, \Phi_\beta^\dagger(\boldsymbol{x}'', t)\Phi_\beta(\boldsymbol{x}'', t)V(|\boldsymbol{x}'' - \boldsymbol{x}|)\Phi_\alpha(\boldsymbol{x}, t). \qquad (3.7)$$

Now we introduce the transformation

$$\Psi_\alpha(\boldsymbol{x}, t) = \exp[-\mathrm{i}\Lambda(\boldsymbol{x}, t)]\Phi_\alpha(\boldsymbol{r}(\boldsymbol{x}, t), t) \qquad (3.8)$$

where

$$\Lambda(\boldsymbol{x}, t) = -\frac{Ft}{\hbar}x + \frac{F^2}{6m\hbar}t^3 \qquad (3.9)$$

and

$$\boldsymbol{r}(\boldsymbol{x}, t) = \boldsymbol{x} - \boldsymbol{\Delta}(t) \qquad (3.10)$$

$$\Delta_x(t) = \frac{F}{2m}t^2 \qquad \Delta_y(t) = 0 \qquad \Delta_z(t) = 0. \qquad (3.11)$$

This transformation is the field theoretical version of the Avron–Herbst transformation defined for the single-particle wavefunction in (2.18). By straightforward computation the following equation is obtained:

$$\left[ i\hbar \frac{\partial}{\partial t} - \tilde{h}_F(x, \nabla) \right] \Psi_\alpha(\boldsymbol{x}, t)$$

$$= \left\{ \exp[-i\Lambda(\boldsymbol{x}, t)] \left( i\hbar \frac{\partial}{\partial t} - \tilde{h}_0(\nabla_r) \right) \Phi_\alpha(\boldsymbol{r}, t) \right\}_{r=r(x,t)=x-\Delta(t)} \tag{3.12}$$

with

$$\tilde{h}_F(\nabla) \equiv \hat{h}_F(x, \partial_x) - \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right). \tag{3.13}$$

In order to derive the equation of motion for the field operator $\Psi_\alpha(\boldsymbol{x}, t)$ we note that the electron–electron interaction term can be expressed as

$$\sum_\beta \int d^3x' \, \Psi_\beta^\dagger(\boldsymbol{x}', t) \Psi_\beta(\boldsymbol{x}', t) V(|\boldsymbol{x} - \boldsymbol{x}'|) \Psi_\alpha(\boldsymbol{x}, t) = \exp[-i\Lambda(\boldsymbol{x}, t)]$$

$$\times \sum_\beta \int d^3x' \, \Phi_\beta^\dagger(\boldsymbol{x}', t) \Phi_\beta(\boldsymbol{x}', t) V(|\boldsymbol{x} - \boldsymbol{x}' - \Delta(t)|) \Phi_\alpha(\boldsymbol{x} - \Delta(t), t)$$

$$= \exp[-i\Lambda(\boldsymbol{x}, t)] \sum_\beta \int d^3x'' \, \Phi_\beta^\dagger(\boldsymbol{x}'' - \Delta(t), t) \Phi_\beta(\boldsymbol{x}'' - \Delta(t), t)$$

$$\times V(|\boldsymbol{x} - \Delta(t) - \boldsymbol{x}'' + \Delta(t)|) \Phi_\alpha(\boldsymbol{x} - \Delta(t), t)$$

$$= \exp[-i\Lambda(\boldsymbol{x}, t)] \left[ \sum_\beta \int d^3r' \, \Phi_\beta^\dagger(\boldsymbol{r}', t) \Phi_\beta(\boldsymbol{r}', t) \right.$$

$$\times V(|\boldsymbol{r}' - \boldsymbol{r}|) \Phi_\alpha(\boldsymbol{r}, t) \Big]_{r=x-\Delta(t)}. \tag{3.14}$$

We now combine (3.12) and (3.14) to obtain the equation of motion for the transformed field operator $\Psi_\alpha(\boldsymbol{x}, t)$

$$i\hbar \frac{\partial}{\partial t} \Psi_\alpha(\boldsymbol{x}, t) = \tilde{h}_F(\nabla) \Psi_\alpha(\boldsymbol{x}, t)$$

$$+ \sum_\beta \int d^3x'' \, \Psi_\beta^\dagger(\boldsymbol{x}'', t) \Psi_\beta(\boldsymbol{x}'', t) V(|\boldsymbol{x}'' - \boldsymbol{x}|) \Psi_\alpha(\boldsymbol{x}, t). \tag{3.15}$$

The equal-time anticommutation relation for the transformed field operators can also be derived

$$[\Psi_\alpha(\boldsymbol{x}, t), \Psi_\beta^\dagger(\boldsymbol{x}', t)]_+ = \exp[-i\Lambda(\boldsymbol{x}, t)] \exp[i\Lambda(\boldsymbol{x}', t)]$$

$$\times [\Phi_\alpha(\boldsymbol{x} - \Delta(t), t), \Phi_\beta^\dagger(\boldsymbol{x}' - \Delta(t), t]_+$$

$$= \exp[-i\Lambda(\boldsymbol{x}, t) + i\Lambda(\boldsymbol{x}', t)] \delta_{\alpha\beta} \delta(\boldsymbol{x} + \Delta(t) - \boldsymbol{x}' - \Delta(t))$$

$$= \delta_{\alpha\beta} \delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{3.16}$$

Thus it has been shown that transformation (3.8) preserves the equal-time anticommutation relation. Results (3.15) and (3.16) also lead to the following Hamiltonian to define the transformed field $\Psi_\alpha(\boldsymbol{x}, t)$

$$H^F = H_0^F + H_{int}^F \tag{3.17}$$

with

$$H_0^F = \sum_\alpha \int \mathrm{d}^3 x \, \Psi_\alpha^\dagger(\boldsymbol{x}, t) \tilde{h}_F(\nabla) \Psi_\alpha(\boldsymbol{x}, t) \tag{3.18}$$

and

$$H_{\mathrm{int}}^F = \tfrac{1}{2} \sum_\alpha \sum_\beta \int \mathrm{d}^3 x \int \mathrm{d}^3 x' \, \Psi_\alpha^\dagger(\boldsymbol{x}, t) \Psi_\beta^\dagger(\boldsymbol{x}', t) V(|\boldsymbol{x} - \boldsymbol{x}'|) \Psi_\beta(\boldsymbol{x}', t) \Psi_\alpha(\boldsymbol{x}, t) \tag{3.19}$$

where we have used the kinetic-energy differential operator given in (3.13). Using the equal-time anticommutation relation (3.16) and the Hamiltonian (3.17), we can derive equation (3.15). In conclusion, transformation (3.8) indeed transforms the electron Schrödinger field without the electric field, $\Phi_\alpha(\boldsymbol{x})$, to that with the electric field, $\Psi_\alpha(\boldsymbol{x}, t)$.

## 4. Time-ordered product of the electron Schrödinger field

In the field theoretical formulation of quantum many-body problems, the Green functions are essential tools to obtain physical quantities from the theory. One of the most widely used Green functions is the ground-state expectation value of the time-ordered product of the field operators [15, 16]

$$G_{\alpha\beta}^0(\boldsymbol{x}, t; \boldsymbol{x}', t') \equiv -\mathrm{i}\langle T\{\Phi_\alpha(\boldsymbol{x}, t)\Phi_\beta^\dagger(\boldsymbol{x}', t')\}\rangle \tag{4.1}$$

which is of particular importance to the perturbative expansion because of the Wick theorem [15, 16]. Here $T\{\ldots\}$ and $\langle\ldots\rangle$ denote the time-ordered product and the ground-state expectation value, respectively. In quantum many-body problems usually the exact ground state is not known except for rare cases of exactly solvable models. Hence, in the conventional approach of the quantum many-body theory, a physically plausible ground state is assumed from the outset. A typical example is Landau's Fermi liquid theory [16, 17].

We separate the properties of the Green function into two parts; one is owing to the ground states and the other is owing to the dynamics of the field operator. In this section we shall be concerned with the dynamics of the Green function, which is determined by the canonical relations for the electron Schrödinger field in the Heisenberg picture. For the aim, we consider the time-ordered product of the field operators

$$\hat{G}_{\alpha\beta}^0(\boldsymbol{x}, t; \boldsymbol{x}', t') \equiv -T\{\Phi_\alpha(\boldsymbol{x}, t)\Phi_\beta^\dagger(\boldsymbol{x}', t')\}. \tag{4.2}$$

The equation for this time-ordered product can be obtained by using the equation of motion for $\Phi_\alpha(\boldsymbol{x}, t)$ given by (3.7)

$$\left[\mathrm{i}\hbar\frac{\partial}{\partial t} - \tilde{h}_0(\nabla)\right]\hat{G}_{\alpha\beta}^0(\boldsymbol{x}, t; \boldsymbol{x}', t') = -\mathrm{i}\hbar\delta_{\alpha\beta}(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$$

$$- \sum_\zeta \int \mathrm{d}^3 x'' \, V(|\boldsymbol{x}'' - \boldsymbol{x}|) T\{\Phi_\zeta^\dagger(\boldsymbol{x}'', t)\Phi_\zeta(\boldsymbol{x}'', t)\Phi_\alpha(\boldsymbol{x}, t)\Phi_\beta^\dagger(\boldsymbol{x}', t')\}. \tag{4.3}$$

Now we proceed to a many-electron system in a uniform electric field. The electron field operators $\Psi_\alpha(\boldsymbol{x})$ and $\Psi_\beta^\dagger(\boldsymbol{x}')$ satisfy the equal-time anticommutation relation

$$[\Psi_\alpha(\boldsymbol{x}), \Psi_\beta^\dagger(\boldsymbol{x}')]_+ = \delta_{\alpha\beta}\delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{4.4}$$

The Hamiltonian of the electrons is assumed to be

$$H^F = H_0^F + H_{\mathrm{int}}^F \tag{4.5}$$

with

$$H_0^F = \sum_\alpha \int d^3x\, \Psi_\alpha^\dagger(x)\tilde{h}_F(\nabla)\Psi_\alpha(x) \tag{4.6}$$

and

$$H_{\text{int}}^F = \tfrac{1}{2}\sum_\alpha \sum_\beta \int d^3x \int d^3x'\, \Psi_\alpha^\dagger(x)\Psi_\beta^\dagger(x')V(|x-x'|)\Psi_\beta(x')\Psi_\alpha(x) \tag{4.7}$$

where we have used the kinetic-energy differential operator defined by (3.13). The above Hamiltonian is equivalent to that defined by (3.17). The Heisenberg picture of a field operator $\Psi_\alpha(x)$ can be expressed in terms of the Hamiltonian as

$$\Psi_\alpha(x,t) \equiv \exp\left(\frac{i}{\hbar}H^F t\right)\Psi_\alpha(x)\exp\left(\frac{-i}{\hbar}H^F t\right). \tag{4.8}$$

The equation of motion for the field operator $\Psi_\alpha(x,t)$ can be obtained by substituting the explicit form of the Hamiltonian (4.5)–(4.7) into (4.8). The equation is

$$i\hbar\frac{\partial}{\partial t}\Psi_\alpha(x,t) = \tilde{h}_F(\nabla)\Psi_\alpha(x,t) + \sum_\beta \int d^3x''\, \Psi_\beta^\dagger(x'',t)\Psi_\beta(x'',t)V(|x''-x|)\Psi_\alpha(x,t)$$

$$\tag{4.9}$$

which is equivalent to (3.15). We define the time-ordered product of the field operators $\Psi_\alpha(x,t)$ and $\Psi_\beta^\dagger(x',t)$,

$$\hat{G}_{\alpha\beta}^F(x,t;x',t') \equiv -T\{\Psi_\alpha(x,t)\Psi_\beta^\dagger(x',t')\}. \tag{4.10}$$

It is straightforward to obtain the equation for $\hat{G}_{\alpha\beta}^F(x,t;x',t')$ by substituting (4.9) into the time derivative of (4.10):

$$\left[i\hbar\frac{\partial}{\partial t} - \tilde{h}_F(\nabla)\right]\hat{G}_{\alpha\beta}^F(x,t;x',t') = -i\hbar\delta_{\alpha\beta}\delta(x-x')\delta(t-t')$$

$$-\sum_\zeta \int d^3x''\, V(|x''-x|)T\{\Psi_\zeta^\dagger(x'',t)\Psi_\zeta(x'',t)\Psi_\alpha(x,t)\Psi_\beta^\dagger(x',t')\}. \tag{4.11}$$

Now we shall show that $\hat{G}_{\alpha\beta}^F(x,t;x',t')$ can be obtained from $\hat{G}_{\alpha\beta}^0(x,t;x',t')$ by a transformation similar to (3.8):

$$\hat{G}_{\alpha\beta}^F(x,t;x't') = \exp[-i\Lambda(x,t) + i\Lambda(x',t')]\hat{G}_{\alpha\beta}^0(r(x,t),t;r(x',t'),t') \tag{4.12}$$

where $\Lambda(x,t)$ and $r(x,t)$ have been defined by (3.9) and (3.10). To clarify the above proposition, we define the right-hand side of (4.12) as

$$\hat{K}_{\alpha\beta}^F(x,t;x',t') \equiv \exp[-i\Lambda(x,t) + i\Lambda(x',t')]\hat{G}_{\alpha\beta}^0(r(x,t),t;r(x',t'),t'). \tag{4.13}$$

Differentiating (4.13) with respect to $t$, we find

$$i\hbar\frac{\partial}{\partial t}\hat{K}_{\alpha\beta}^F(x,t;x',t') = \hbar\frac{\partial\Lambda(x,t)}{\partial t}\hat{K}_{\alpha\beta}^F(x,t;x',t')$$

$$-i\hbar\exp[-i\Lambda(x,t)+i\Lambda(x',t')]\delta(t-t')$$

$$\times[\Phi_\alpha(x-\Delta(t),t),\Phi_\beta^\dagger(x'-\Delta(t'),t')]$$

$$-\exp[-i\Lambda(x,t)+i\Lambda(x',t')]T\left\{i\hbar\frac{\partial\Phi_\alpha(x-\Delta(t),t)}{\partial t}\Phi_\beta^\dagger(x'-\Delta(t'),t')\right\}$$

$$\tag{4.14}$$

where the first term on the right-hand side simply gives

$$\hbar \frac{\partial \Lambda(x, t)}{\partial t} \hat{K}_{\alpha\beta}^{F}(x, t; x', t') = \left(-Fx + \frac{F^2}{2m}t^2\right) \hat{K}_{\alpha\beta}^{F}(x, t; x', t') \qquad (4.15)$$

and the time derivative in the third term on the right-hand side of (4.14) gives

$$\frac{\partial \Phi_\alpha(x - \Delta(t), t)}{\partial t} = \sum_{k=1}^{3} \left(\frac{\partial \Phi_\alpha(r, t)}{\partial r_k}\right)_t \left(\frac{\partial r_k}{\partial t}\right)_x + \left(\frac{\partial \Phi_\alpha(r, t)}{\partial t}\right)_r$$

$$= \frac{-Ft}{m} \frac{\partial \Phi_\alpha(x - \Delta(t), t)}{\partial x} + \left(\frac{\partial \Phi_\alpha(r, t)}{\partial t}\right)_r. \qquad (4.16)$$

Using (3.7) for the second term on the right-hand side of (4.16), we obtain

$$i\hbar \frac{\partial \Phi_\alpha(x - \Delta(t), t)}{\partial t} = \frac{-i\hbar Ft}{m} \left(\frac{\partial \Phi_\alpha(x - \Delta(t), t)}{\partial x}\right)_t - \frac{\hbar^2}{2m} \nabla_x^2 \Phi_\alpha(x - \Delta(t), t)$$

$$+ \sum_\beta \int \mathrm{d}^3 x'' \, \Phi_\beta^\dagger(x'', t) \Phi_\beta(x'', t) V(|x'' - x + \Delta(t)|) \Phi_\alpha(x - \Delta(t), t). \qquad (4.17)$$

This result further yields

$$\exp[-i\Lambda(x, t)]i\hbar \frac{\partial \Phi_\alpha(x - \Delta(t), t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla_x^2 (\exp[-i\Lambda(x, t)]\Phi_\alpha(x - \Delta(t), t))$$

$$- \frac{F^2 t^2}{2m} \exp[-i\Lambda(x, t)]\Phi_\alpha(x - \Delta(t), t)$$

$$+ \sum_\beta \int \mathrm{d}^3 x'' \, \Phi_\beta^\dagger(x'', t) \Phi_\beta(x'', t) V(|x'' - x + \Delta(t)|)$$

$$\times \exp[-i\Lambda(x, t)]\Phi_\alpha(x - \Delta(t), t)$$

$$= -\frac{\hbar^2}{2m} \nabla_x^2 \Psi_\alpha(x; t) - \frac{F^2 t^2}{2m} \Psi_\alpha(x, t)$$

$$+ \sum_\beta \int \mathrm{d}^3 x'' \, \Psi_\beta^\dagger(x'', t) \Psi_\beta(x'', t) V(|x'' - x|) \Psi_\alpha(x, t). \qquad (4.18)$$

For the derivation of the electron–electron interaction term in the last line of (4.18), we used (3.14). Now, combining equations (4.14)–(4.18) we obtain the final result:

$$i\hbar \frac{\partial}{\partial t} \hat{K}_{\alpha\beta}^{F}(x, t; x', t') = \tilde{h}_F(\nabla) \hat{K}_{\alpha\beta}^{F}(x, t; x', t') - i\hbar \delta_{\alpha\beta} \delta(x - x') \delta(t - t')$$

$$+ \sum_\zeta \int \mathrm{d}^3 x'' \, V(|x'' - x|) T\{\Psi_\zeta^\dagger(x'', t)\Psi_\zeta(x'', t)\Psi_\alpha(x, t)\Psi_\beta^\dagger(x', t')\} \qquad (4.19)$$

which is the expected equation of motion for the time-ordered product $\hat{G}_{\alpha\beta}^{F}$ given by (4.11). Thus, we have proved that the time-ordered product $\hat{K}_{\alpha\beta}^{F}(x, t; x', t')$ and $\hat{G}_{\alpha\beta}^{F}(x, t; x', t')$ satisfy the same equation.

## 5. Concluding remarks

We have shown an extension of the Avron–Herbst transformation to the second quantized Schrödinger field for an interacting electron gas. Although the calculation in this paper is restricted to the electron field, i.e. the field operator satisfying the equal-time

anticommutation relation such as (3.2), the entire calculation in sections 3 and 4 can hold for the boson field satisfying the equal-time commutation relation. That is, equations (3.15) and (4.19) can also hold for the Schrödinger field satisfying the boson-type commutation relation. Therefore, it is straightforward to apply the present formulation to a boson system such as liquid He$^4$ to investigate the effect of, for instance, the gravitational field.

Regarding the time-ordered product discussed in section 4, we remark that a practical application is possible by introducing a physically acceptable ground state for the Hamiltonian (4.5). One of such possible ground states may be a Fermi-liquid-type ground state whose Fermi sphere has a shifted centre in the momentum space. The shift takes place owing to the acceleration of electrons by an electric field. The magnitude of the shift must be determined self-consistently by taking account of the effects of dissipation. Once we introduce such a ground state, the transformation derived in section 4 can be applied to the Green functions. Such an application of the Avron–Herbst transformation as well as a further extension to the density–density response function will be discussed in a forthcoming paper.

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## References

[1]  Blatt F J 1968 *Physics of Electric Conduction in Solids* (New York: McGraw-Hill)
[2]  Meaden G T 1965 *Electrical Resistance of Metals* (New York: Plenum)
[3]  Toyoda T 1989 *Phys. Rev.* A **39** 2659
[4]  Zubarev D N 1974 *Nonequilibrium Statistical Thermodynamics* (New York: Consultants Bureau)
[5]  Ziman J M 1960 *Electrons and Phonons* (Oxford: Oxford University Press)
[6]  Wannier G H 1960 *Phys. Rev.* **117** 432
     Wannier G H 1969 *Phys. Rev.* **181** 1364
[7]  Wannier G H and Fredkin D R 1962 *Phys. Rev.* **125** 1910
[8]  Zak J 1968 *Phys. Rev. Lett.* **20** 1477
     Zak J 1969 *Phys. Rev.* **181** 1366
[9]  Moyer C A 1973 *Phys. Rev.* B **7** 5025
[10] Fritsche L 1966 *Phys. Status Solidi* **13** 487
[11] Roy C L and Mahapatra P K 1980 *J. Phys. C: Solid State Phys.* **13** 5365
[12] Avron J E and Herbst I W 1977 *Commun. Math. Phys.* **52** 239
[13] Landau L D and Lifshitz E M 1958 *Quantum Mechanics* (Oxford: Pergamon)
[14] Bleistein N and Handelsman R A 1975 *Asymptotic Expansions of Integrals* (New York: Dover)
[15] Fetter A L and Walecka J D 1971 *Quantum Theory of Many-particle Systems* (New York: McGraw-Hill)
[16] Abrikosov A A, Gor'kov L P and Dzyaloshinskii I Ye 1965 *Quantum Field Theoretical Methods in Statistical Physics* 2nd edn (Oxford: Pergamon)
[17] Toyoda T 1987 *Ann. Phys., NY* **173** 226